

The University of Chicago

THE PROBLEM OF LAGRANGE WITH
FINITE SIDE CONDITIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1933

By

JULIA WELLS BOWER

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THE PROBLEM OF LAGRANGE WITH FINITE SIDE CONDITIONS

INTRODUCTION

The problem of Lagrange with finite side conditions in the calculus of variations as stated by Bliss (8, p. 703)¹ is that of finding among the arcs

$$(1) \quad y_1 = y_1(x) \quad (x_1 \leq x \leq x_2; 1 = 1, \dots, n)$$

joining two given points 1 and 2 in the space of points $(x, y_1, \dots, y_n)$ and satisfying a set of equations of the form

$$(2) \quad \phi_\alpha(x, y_1, \dots, y_n) = 0 \quad (\alpha = 1, \dots, m < n)$$

one which minimizes the integral

$$I = \int_{x_1}^{x_2} f(x, y_1, \dots, y_n; y_1', \dots, y_n') dx.$$

Two methods have been suggested for the development of a complete theory for the solution of this problem. The first is to transform it by differentiation of the side conditions (2) into a problem of Lagrange with differential equations as side conditions. The second, by solution of equations (2) for m of the variables $y_1, \dots, y_n$ in terms of the remaining $n - m$, replaces it by an equivalent simple problem of the calculus of variations. For the last of these new problems the complete solution theory is well known. The first one has however abnormalities which have not been completely discussed in the literature. Although the possibility of solution by each of these methods has long

¹The first numbers in the parentheses refer to the list of references at the end of the paper.

been recognized, the application of them to the problem has not until now been made in detail. In this paper both procedures are used and by each of them a complete theory for the problem is deduced.

When the first method is employed it is found (8, p. 704) that the relations

$$\sum_{i=1}^n \phi_{\alpha y_i} \eta_i = \text{constant}$$

hold among the variations $\eta_i(x)$. Hadamard (3, § 207), Mayer (1, p. 80 footnote), and Bolza in his "Vorlesungen" (p. 582), all note that these conditions make the new problem abnormal. Bolza in a later paper (4, p. 444) shows that the problem becomes normal when the end conditions are reduced (4, pp. 432, 444) and normality relative to the new end conditions is properly defined. In the present discussion in accordance with the remark of Bolza m of the variables $y_1, \dots, y_n$ are left undetermined at the first end-point, and the problem becomes a problem of Lagrange with differential equations as side conditions and first end-point variable. In this form it is referred to as the transformed problem. The admissible arcs for it are seen to be normal and normal on every sub-interval relative to reduced end conditions. But they are not normal on every sub-interval relative to fixed end-points (8, p. 687), as is required in the theories hitherto given for problems with variable end-points.

In Chapter I a complete theory for the solution of the problem of Lagrange with finite side conditions is deduced with the help of the transformed problem. The first variation of the integral I yields the well-known Necessary Condition I with its corollaries. The derivation by Graves of the Necessary Condition of Weierstrass for the problem of Bolza for an arc normal with

respect to the end conditions (see 13) can readily be applied here, but the special form of the normality relations makes possible the simpler proof given in Section 4. The Necessary Condition of Mayer becomes a condition involving focal points of the initial manifold. Certain modifications are required in adapting the analytic proof of the Mayer condition for the problem of Lagrange to this problem because of the fact already noted that admissible arcs for it are not normal on any sub-interval relative to fixed end-points. For this same reason the sufficient conditions as derived for the problem of Lagrange with fixed end-points by Bliss (8, Chapter IV) and Morse (11), for the problem of Bolza with variable end-points by Bliss (12, pp. 267-273), and for the problem of Mayer by Bliss and Hestenes (14, pp. 317-325), are not applicable here. The procedure followed in Section 7 in the development of such conditions is, however, very simple and leads to results similar to those for the problem of Lagrange with fixed end-points.

In Chapter II the theory for the problem of Lagrange with finite side conditions is deduced by the second method suggested above. It has been customary to make the unsymmetrical assumption that a particular determinant of order m of the matrix $\| \Phi_{\alpha y_i} \|$ is different from zero along the arc E_{12} . In the present case such a situation is avoided by a method of adjunction due to Bliss. A set of $n - m$ suitably chosen equations

$$(3) \quad \phi_r(x, y_1, \dots, y_n) = z_r \quad (r = m+1, \dots, n)$$

is adjoined to the side conditions (2) and the resulting system is solved for the variables $y_1, \dots, y_n$ in terms of the functions $z_{m+1}, \dots, z_n$. A new simple problem of the calculus of variations,

completely equivalent to the original one, is then set up in $(x, z_{m+1}, \dots, z_n)$ -space. The contribution of this paper for the problem in this form is the interpretation for the original problem in $(x, y_1, \dots, y_n)$ -space of the well-known necessary and sufficient conditions for a minimum in the simple problem in $(x, z_{m+1}, \dots, z_n)$ -space where the auxiliary functions (3) are eliminated from the statement of the conditions. Results identical with those in Chapter I are obtained.

CHAPTER I

THE PROBLEM OF LAGRANGE WITH FINITE SIDE CONDITIONS AS A SIMILAR PROBLEM WITH DIFFERENTIAL EQUATIONS AS SIDE CONDITIONS

1. Introduction. Let E_{12} with equations (1) be a minimizing arc for the problem of Lagrange with finite side conditions. It is presupposed that

(1) the functions $y_i(x)$ defining E_{12} are continuous on the interval x_1x_2 , and the interval can be divided into a finite number of parts on each of which the functions have continuous derivatives;¹

(2) in a neighborhood R of the values (x, y, y') on the arc E_{12} the functions f and ϕ_α have continuous derivatives up to and including those of third and fourth orders respectively;

(3) at every element (x, y, y') on E_{12} the $m \times n$ dimensional matrix $\|\phi_{\alpha y_i}\|$ has rank m , and in particular at the point 1 the determinant $|\phi_{\alpha y_\delta}|$, ($\delta = 1, \dots, m$), is different from zero.

As indicated in the preceding section, this problem may be transformed by differentiation of the side conditions and reduction of the end conditions (8, p. 703), into the problem considered in this chapter, namely, that of finding in the class of arcs

$$(1:1) \quad y_1 = y_1(x) \quad (x_1 \leq x \leq x_2; 1 = 1, \dots, n)$$

satisfying the differential equations

$$(1:2) \quad \phi'_\alpha \equiv \phi_{\alpha x} + \phi_{\alpha y_i} y'_i = 0 \quad (\alpha = 1, \dots, m < n)$$

¹Throughout this paper the notation of Bliss (8, pp. 675-678) is used.

and end conditions

$$\psi_{\mu}[x_1, y_1(x_1), x_2, y_1(x_2)] = 0 \quad (\mu = 1, \dots, p = 2n-m+2),$$

which for the problem with fixed end-points takes the specific form

$$(1:3) \quad x_1 - \alpha_1 = y_1(x_1) - \beta_{11}, \quad x_2 - \alpha_2 = y_1(x_2) - \beta_{12} = 0 \\ (r = m + 1, \dots, n),$$

one which minimizes the integral

$$I = \int_{x_1}^{x_2} f(x, y, y') dx.$$

The new problem is equivalent to the original one in a sufficiently small neighborhood of the minimizing arc since admissible arcs for it satisfy the equations (1:2) and the last n equations (1:3). Hence they satisfy also the equations (2) which with end conditions (1:3) suffice to determine the unprescribed $y_{\alpha}(x_1)$ as coordinates $\beta_{\alpha 1}$ on E_{12} . From properties (1), (2) and (3) above, it follows that the hypotheses of Bliss for the problem of Lagrange (8, p. 675) are satisfied by the functions y_1 , f and ϕ'_{α} of the new problem, except for the weakening of his assumption (b) by substitution of "third order" for "fourth order" and the inclusion of the non-singularity of determinant $|\phi_{\alpha y_{\delta}}(x_1)|$ in his assumption (c). This last is not an essential restriction since it can be brought about by renumbering the variables y_1 .

At every point x' on E_{12} the matrix $\|\phi_{\alpha y_i}\|$ has a non-vanishing determinant $|\phi_{\alpha y_{\tau}}|$, where τ ranges over m suitably selected integers of the set $1, \dots, n$. If t varies over the remaining $n - m$ integers, and if end conditions for the interval $x'x''$ ($x_1 \leq x' < x'' \leq x_2$) are defined by equations

$$(1:4) \quad x' - \alpha' = y_t(x') - \beta_t' = x'' - \alpha'' = y_1(x'') - \beta_1'' = 0,$$

then the arc E_{12} is said to be normal on the sub-interval $x'x''$ in case there exist for it $p = 2n - m + 2$ sets of admissible variations $\xi_\sigma', \xi_\sigma'', \eta_{i\sigma}, (\sigma = 1, \dots, p)$, for which the determinant

$$\begin{vmatrix} \xi_\sigma' \\ \eta_{t\sigma}(x') \\ \xi_\sigma'' \\ \eta_{i\sigma}(x'') \end{vmatrix}$$

is different from zero. By the method of Bliss for extremal arcs (8, p. 704) it can be shown that admissible arcs for the transformed problem are normal on every sub-interval relative to the conditions (1:4), as well as relative to the end conditions $\psi_u = 0$.

2. Necessary Conditions from the First Variation. In this section conditions imposed on the minimizing arcs by the first variation of the integral I will be discussed with the help of the transformed problem. A modification by Bliss and Hestenes (15, p.) of the First Necessary Condition for the problem of Lagrange (8, p. 692) yields the following theorem.

NECESSARY CONDITION I. For every minimizing arc for the transformed problem there exist a set of constants c_0, c_1 ($i = 1, \dots, n$) and a function

$$F(x, y, y', \lambda) = f + \lambda_\alpha \phi_\alpha' \quad (\alpha = 1, \dots, m)$$

such that equations

$$(2:1) \quad F - y_1' F_{y_1'} = \int_{x_1}^x F_x dx + c_0 \quad (i = 1, \dots, n)$$

$$(2:2) \quad F_{y_1'} = \int_{x_1}^x F_{y_1'} dx + c_1$$

hold at every point of E_{12} . The multipliers λ_α are unique and are continuous except possibly at values of x defining corners of E_{12} . Furthermore the end-values of E_{12} must be such that the transversality conditions

$$(2:3) \quad F_{y_\alpha}'(x_1) = 0$$

are satisfied.

The simplicity of equations (2:3) is due to the specific form of the end-conditions (1:3), and the uniqueness of the functions λ_α is a consequence of the fact (8, p. 704) that minimizing arcs are normal with respect to relations (2:2) and (2:3).

Three corollaries follow as usual from Necessary Condition I.

COROLLARY 1. THE EULER-LAGRANGE MULTIPLIER RULE. On every sub-arc between corners of a minimizing arc E_{12} the differential equations

$$(2:4) \quad (d/dx) F_{y_1}' = F_{y_1}, \quad \phi_\alpha'(x, y, y') = 0$$

must be satisfied.

COROLLARY 2. THE CORNER CONDITION. At every corner of a minimizing arc E_{12} the functions

$$F - y_1' F_{y_1}' = f - y_1' f_{y_1}' + \lambda_\alpha \phi_{\alpha x},$$

$$F_{y_1}' = f_{y_1}' + \lambda_\alpha \phi_{\alpha y_1}$$

are continuous.

It is readily seen that this second corollary is equivalent to the condition that at every corner of a minimizing arc the set of functions $1, \lambda_\alpha(x - 0) - \lambda_\alpha(x + 0)$ satisfies the linear equations whose coefficients are the rows of the matrix

$$(2:5) \quad \left\| \begin{array}{cc} f - y_1' f_{y_1}' & \left| \begin{array}{c} x - 0 \\ x + 0 \end{array} \right. \phi_{\alpha x} \\ f_{y_1}' & \left| \begin{array}{c} x - 0 \\ x + 0 \end{array} \right. \phi_{\alpha y_i} \end{array} \right\| .$$

COROLLARY 3. THE DIFFERENTIABILITY CONDITION. Near a point of a minimizing arc E_{12} at which the determinant

$$R = \left| \begin{array}{cc} f_{y_1' y_k'} & \phi_{\alpha y_i} \\ \phi_{\beta y_k} & 0 \end{array} \right| \quad \begin{array}{l} (k = 1, \dots, n; \\ \beta = 1, \dots, m) \end{array}$$

is different from zero the functions $y_1(x)$ defining E_{12} have continuous derivatives of second order at least and the multipliers $\lambda_\alpha(x)$ have at least continuous first derivatives.

An arc along which the determinant R is different from zero is called non-singular. The theorem below is analogous to one derived by Bliss for extremal arcs (8, p. 705).

THEOREM 2:1. For every non-singular minimizing arc E_{12} for the problem of Lagrange with finite side conditions there exists a function

$$G = f + \Lambda_\alpha \phi_\alpha$$

such that equations

$$(2:6) \quad (d/dx) G_{y_1} = G_{y_1}, \quad \phi_\alpha(x, y) = 0$$

are satisfied on every sub-arc between corners of E_{12} . The multipliers Λ_α have the values $-\lambda'_\alpha$ and are unique. They are continuous except possibly at corner values of x . Moreover at each corner of E_{12} the matrix (2:5) has rank less than $m + 1$.

It is true conversely that whenever a non-singular minimizing arc E_{12} for the original problem satisfies the conditions of Theorem 2:1 it is also a non-singular minimizing arc for the

transformed problem along which Corollaries 1 and 2 hold. For if the functions λ_α are defined as indefinite integrals of Λ_α , possibly discontinuous at the corners of E_{12} , and if at each corner their discontinuities are set equal to constants k_α which together with the number 1 form solutions of the system of linear equations having rows of matrix (2:5) as coefficients, and if finally their end-values at 1 are prescribed by transversality conditions (2:3) for the transformed problem, then the multipliers λ_α are uniquely determined and the minimizing arc can be shown to satisfy Necessary Condition I and its Corollaries 1 and 2.

3. The Extremals. In Section 6 of his paper on the problem of Lagrange (8, pp. 684-687), Bliss gives two methods of imbedding a non-singular extremal arc in a $2n$ -parameter family of extremals. When the procedure first suggested is applied to the transformed problem, equations (2:4) are replaced by the equivalent system

$$(3:1) \quad \begin{aligned} (d/dx)F_{y_1'} - F_{y_1} &= f_{y_1'}x + f_{y_1'}y_k y_k' + f_{y_1'}y_k y_k'' \\ &+ \lambda_\alpha' \phi_\alpha y_i - f_{y_1} = 0 \quad (i, k = 1, \dots, n), \end{aligned}$$

$$(3:2) \quad (d/dx) \phi_\alpha' = \phi_{\alpha x}' + \phi_{\alpha y_k}' y_k' + \phi_{\alpha y_k} y_k'' = 0 \quad (\alpha = 1, \dots, m),$$

$$(3:3) \quad \phi_\alpha'[x_1, y(x_1), y'(x_1)] = 0.$$

Near E_{12} equations (3:1) and (3:2) can be solved for y_k'' , λ_α' and are equivalent to a set of the following form

$$dy_k/dx = y_k', \quad dy_k'/dx = Y_k(x, y, y'), \quad d\lambda_\alpha/dx = L_\alpha(x, y, y').$$

The integrals of the first $2n$ of these contain $2n$ arbitrary constants of which m may be eliminated by relations (3:3). Call the remaining parameters $a_1, \dots, a_{2n-m}$. The solution of the last m equations for the multipliers λ_α introduces m new constants of

integration, $a_{2n-m+1}, \dots, a_{2n}$. Hence the non-singular extremal arc E_{12} is a member of a $2n$ -parameter family of extremals

$$(3:4) \quad \begin{aligned} y_1 &= y_1(x, a_1, \dots, a_{2n-m}), \\ \lambda_\alpha &= \lambda_\alpha(x, a_1, \dots, a_{2n-m}) + a_{2n-m+\alpha}. \end{aligned}$$

By use of definitions - $\Lambda_\alpha = \lambda'_\alpha$ and conditions

$$\phi_\alpha[x_1, y(x_1)] = 0$$

the family (3:4) yields a $(2n - 2m)$ -parameter family of extremal arcs for the original problem

$$y_1 = y_1(x, b_1, \dots, b_{2n-2m}), \quad \Lambda_\alpha = \Lambda_\alpha(x, b_1, \dots, b_{2n-2m})$$

satisfying the Euler-Lagrange equations in the form (2:6).

Processes in the latter part of the reference above show that a non-singular extremal arc E_{12} for the transformed problem is also imbedded in a $2n$ -parameter family of extremals

$$\begin{aligned} y_1 &= y_1(x, a_1, \dots, a_n, b_1, \dots, b_n), \\ v_1 &= v_1(x, a_1, \dots, a_n, b_1, \dots, b_n) \end{aligned}$$

in which functions v_1 are defined by equations $v_1 = F_{y_1}$, and the parameters a_1, b_1 are the initial values of variables y_1, v_1 , respectively, for a fixed x on E_{12} . The determinant

$$\begin{vmatrix} y_{1a_k} & y_{1b_k} \\ v_{1a_k} & v_{1b_k} \end{vmatrix}$$

is different from zero for every value of x (8, p. 727).

4. The Necessary Conditions of Weierstrass and Clebsch.

The proof by Graves of the Necessary Condition of Weierstrass for the problem of Bolza (see 13) for a minimizing arc normal relative to the end conditions is applicable here when the function G in

the notation of Graves is set equal to zero. For the problem considered in this chapter a simpler proof can be made similar to that of Bliss (8, p. 716) for an arc normal on every sub-interval according to the usual definition (8, p. 687), and similar also to that of Boyce (10, p. 12) for the problem of Mayer at points 3 for which the sub-arc E_{32} is normal relative to its end conditions.

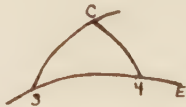
Let the minimizing arc E_{12} have equations

$$y_i = y_i(x) \quad (x_1 \leq x \leq x_2; i = 1, \dots, n).$$

Through an admissible set (x_3, y_{13}, Y'_{13}) at the arbitrary point 3 of E_{12} there is always an admissible arc

$$(C) \quad y_i = Y_i(x) \quad (x_3 \leq x \leq x_3 + h).$$

For, whenever a properly chosen set of $n - m$ of the functions Y through their prescribed initial values of Y and Y' at the point 3 is given, the remaining m functions Y are determined with the given initial values at 3 by equations $\phi_\alpha = 0$. Restrict h of the interval $x_3, x_3 + h$ on which C is defined so that the determination of these remaining m coordinates at each point is unique. It will be convenient to designate the arbitrarily chosen functions by the notations $Y_t(x)$ where the range of the subscript t is described in Section 1.



Since the sub-arc E_{34} of E_{12} , so small that it contains no corners, is normal in the sense of Section 1, there is (8, p. 691) a p -parameter family of admissible arcs ($p = 2n - m + 2$)

$$y_1 = y_1(x, b_1, \dots, b_p) \quad [x_3(b_1, \dots, b_p) \leq x \leq x_4(b_1, \dots, b_p)]$$

$$x_\lambda(b_1, \dots, b_p) = x_\lambda + b_\sigma \xi_1^\sigma \quad (\lambda = 3, 4; \sigma = 1, \dots, p)$$

containing E_{12} for parameter values $(b_1, \dots, b_p) = (0, \dots, 0)$, and having p sets of admissible variations ξ_3^σ , ξ_4^σ , $\eta_i^\sigma(x)$ for which the determinant

$$(4:1) \quad \begin{vmatrix} \xi_3^\sigma \\ \eta_t^\sigma(x_3) \\ \xi_4^\sigma \\ \eta_i^\sigma(x_3) \end{vmatrix}$$

is different from zero. The p equations

$$y_t(x_5, b_1, \dots, b_p) = Y_t(x_5),$$

$$y_1(x_4, b_1, \dots, b_p) = y_{14},$$

$$x_\lambda(b_1, \dots, b_p) = x_\lambda + b_\sigma \xi_1^\sigma$$

have the initial solution $(x_5, b_1, \dots, b_p) = (x_3, 0, \dots, 0)$ at which their functional determinant with respect to $b_1, \dots, b_p$ is the determinant (4:1) different from zero. They have therefore p solutions $b_\mu = B_\mu(x_5)$ which vanish for $x_5 = x_3$. The family defined by the equations

$$y_1 = y_1[x, B_1(x_5), \dots, B_p(x_5)] = y_1(x, x_5)$$

$$x_\lambda[B_1(x_5), \dots, B_p(x_5)] = x_\lambda + B_\sigma(x_5) \xi_1^\sigma$$

is now a one-parameter family of admissible arcs joining curve C to point 4. If $I(E_{12})$ is to be a minimum, the sum

$$I(C_{35}) + I(E_{54}) = \int_{x_3}^{x_5} f(x, Y, Y') dx + \int_{x_5}^{x_4} f[x, y(x, x_5), y'(x, x_5)] dx$$

must have its derivative with respect to $x_5 \stackrel{>}{=} 0$ at $x_5 = x_3$. This derivative (8, p. 717) is the value at x_3 of the expression

$$(4:2) \quad \begin{aligned} E(x, y, y', Y', \lambda) &= F(x, y, Y', \lambda) - F(x, y, y', \lambda) \\ &\quad - (Y_1' - y_1')F_{y_1'}(x, y, y', \lambda). \end{aligned}$$

Because of the linearity of the relations $\phi'_\alpha = 0$ in the variables y_1' , the function (4:2) reduces further to

$$(4:3) \quad \begin{aligned} E(x, y, y', Y') &= f(x, y, Y') - f(x, y, y') \\ &\quad - (Y_1' - y_1')f_{y_1'}(x, y, y'). \end{aligned}$$

THE NECESSARY CONDITION OF WEIERSTRASS. At each element
(x, y, y') of a minimizing arc the inequality

$$E(x, y, y', Y') \geq 0$$

must be satisfied for every admissible set (x, y, Y') $\neq$ (x, y, y'), where E is given by formula (4:3). The admissibility required is
for the transformed problem and implies the equations

$$\phi_{\alpha x} + \phi_{\alpha y_i} Y_1' = 0 \quad (\alpha = 1, \dots, m; i = 1, \dots, n).$$

The linearity of the side conditions $\phi'_\alpha = 0$ in the variables y_1' also makes the Necessary Condition of Clebsch for the problem of Lagrange with finite side conditions, as deduced by means of the transformed problem (8, p. 718), simpler in form than that for the problem of Lagrange with differential equations as side conditions.

THE NECESSARY CONDITION OF CLEBSCH. At each element
(x, y, y') of a minimizing arc the inequality

$$f_{y_1' y_k'} \pi_1 \pi_k \geq 0$$

must be satisfied for every set ($\pi_1, \dots, \pi_n$) $\neq$ (0, ..., 0)

which satisfies the equations

$$\phi_{\alpha y_i} \pi_i = 0.$$

It should be noted that although the derivatives ϕ_{α}' appear in the statement of the transformed problem they do not do so in the results above which are immediately applicable to minimizing arcs for the original problem.

5. The Necessary Condition of Mayer. A non-singular extremal arc E_{12} for the transformed problem, whose end-values satisfy $\gamma_{\mu} = 0$, has its initial end-point on the manifold M determined by the conditions

$$(5:1) \quad x_1 = \alpha, \quad , \quad y_r = \beta_{r1} \quad (r = m + 1, \dots, n).$$

In this section a focal point of M on E_{12} will be defined, and it will be shown by both geometric and analytic methods that if E_{12} minimizes the integral I , then there can be no such focal point on it between 1 and 2. It will moreover be evident that this result implies a corresponding Mayer condition for the original problem.

Boyce has derived a determinant (10, p. 31) whose roots give the focal points of the terminal manifold of a particular Mayer problem with second end-point variable. The non-singular extremal arc on which these points lie is imbedded in a $2n$ -parameter family of extremals similar to the last family of Section 3 of this paper when the parameters are specialized to be the values of the functions y_1 and v_1 at x_2 on E_{12} . For the particular form of the end-values and transversality conditions of the transformed problem, and for general parameters, his procedure yields the following definition: a value $x_3 \neq x_1$ is said to define a focal point 3 of the manifold (5:1) on the non-singular extremal arc E_{12}

if it is a root for $a_1 = a_{10}$, $b_1 = b_{10}$ of the determinant

$$(5:2) \quad \Delta(x, x_1, a, b) = \begin{vmatrix} y_{1a_k}(x, a, b) & y_{1b_k}(x, a, b) \\ v_{\alpha a_k}(x_1, a, b) & v_{\alpha b_k}(x_1, a, b) \\ y_{ra_k}(x_1, a, b) & y_{rb_k}(x_1, a, b) \end{vmatrix}$$

belonging to a $2n$ -parameter family of extremals

$$\begin{aligned} y_1 &= y_1(x, a_1, \dots, a_n, b_1, \dots, b_n), \\ \lambda_\alpha &= \lambda_\alpha(x, a_1, \dots, a_n, b_1, \dots, b_n) \end{aligned}$$

which contains E_{12} for parameter values a_{10} , b_{10} .

A one-parameter family of extremal arcs having an envelope, as described in the following theorem, may be constructed by the argument of Bliss in the derivation of an analogous theorem for the problem of Lagrange (8, p. 720).

Let E_{12} be an extremal arc along which the determinant R is different from zero, and let 3 be a focal point of M on E_{12} , according to the definition given above, at which the derivative Δ_x of the determinant (5:2) is different from zero. Then there exists a one-parameter family of extremal arcs

$$(5:3) \quad y_1 = y_1(x, t), \quad \lambda_\alpha = \lambda_\alpha(x, t)$$

satisfying the initial end-point and transversality conditions, containing E_{12} for parameter value $t = 0$, and having an envelope D which touches E_{12} at point 3. The functions y_1 , y_{ix} , λ_α , and the function $x(t)$ defining D have continuous derivatives in a neighborhood of the values x , t belonging to the arc E_{12} .

This family has the important property of

THE ENVELOPE THEOREM. If the envelope of the one-parameter family of extremal arcs (5:3) has a branch projecting backward from 3 toward the manifold M as shown in the figure, then



for every position of the point 4 on D near to and preceding 3
the composite arc $K_{52} = E_{54} + D_{43} + E_{32}$ is an admissible arc satis-
fying equations $\phi_\alpha = 0$. For such an arc $I(K_{52}) = I(E_{12})$.

The first conclusion is a direct consequence of the fact that since D is tangent at each point to an extremal of the family, in particular at 3 to E_{12} , it satisfies the equations $\phi'_\alpha = 0$, and hence also the equations $\phi_\alpha = 0$ since the functions ϕ_α vanish at the point 3. It follows readily that every composite arc K_{52} also satisfies these equations. For each of these arcs the composite curve $E_{54} + D_{43}$ gives to the integral I a constant value since the t-derivative of the function $I(E_{54} + D_{43})$ vanishes (8, p. 721). Hence the final conclusion is justified. It is clear that the extremal arcs considered in this theorem are also extremals for the original problem, and that for those lying in the region in which the original and transformed problems are equivalent the first end-points all fall at 1 on E_{12} .

If the non-singular extremal arc E_{12} of the Envelope Theorem is a minimizing arc, then each composite curve $E_{54} + D_{43} + E_{32}$ is also a minimizing arc, since it gives the integral I the same value as E_{12} . It is further a normal extremal since it is admissible and non-singular. At x_2 the unique set of multipliers 1, λ_α and the functions y_1, y_1' which determine it are obviously those of E_{12} , with which it must then coincide because the extremal E_{12} is the only one having these values of $y_1, y_1', \lambda_\alpha$ at $x = x_1$. But this coincidence is not possible (8, p. 722). A fourth necessary condition, evidently applicable to extremals for the original

problem, has thus been established by geometric proof.

THE NECESSARY CONDITION OF MAYER. Let E_{12} be an extremal arc along which the determinant R is different from zero. If E_{12} is a minimizing arc for the problem of Lagrange with finite side conditions, or for the equivalent transformed problem, then no focal point ζ of M on E_{12} can lie between 1 and 2.

The proof of this condition given above is geometric in character and is valid only for focal points which are contact points of envelope arcs as described in the preceding paragraphs. The analytic proof of this condition requires a new definition of a focal point in terms of relations deduced from the second variation of the integral I . The theorem below can be verified by the usual method of computing the second variation (8, p. 723; see also 12, p. 266 and 7, p. 187).

THEOREM 5:1. Along an extremal arc E_{12} which satisfies the end conditions $\psi_\mu = 0$ and the First Necessary Condition the second variation of the integral I is expressible in the form

$$I''(0) = \int_{x_1}^{x_2} 2\omega(x, \eta, \eta') dx + \left[(F_x + y_1' F_{y_1}) \xi^2 + 2F_{y_1} \eta_1 \xi \right]_{x_1}^{x_2},$$

where

$$2\omega = F_{y_1 y_k} \eta_1 \eta_k + 2F_{y_1 y_k'} \eta_1 \eta_k' + F_{y_1' y_k'} \eta_1' \eta_k'.$$

If E_{12} is a minimizing arc for the original or the transformed problem, then for every set of admissible variations $\xi, \eta_1(x)$ whose end values satisfy the equations

$$(5:4) \quad \Psi_\mu(\xi, \eta) \equiv \xi_1 = \eta_\mu(x_1) = \xi_2 = \eta_1(x_2) = 0 \\ (\mu = 1, \dots, p = 2n - m + 2)$$

the condition

$$(5:5) \quad I''(0) = \int_{x_1}^{x_2} 2\omega(x, \eta, \eta') dx \geq 0$$

must be satisfied.

In connection with the end conditions (5:4) it is evident that the values $\eta_\alpha(x_1)$ also vanish since the equations of variation

$$(5:6) \quad \Phi'_\alpha(x, \eta, \eta') = \Phi_{\alpha y_i} \eta'_i + \Phi_{\alpha y_i'} \eta'_i = 0.$$

are equivalent to the system

$$\Phi_{\alpha y_i}(x) \eta_i(x) = \Phi_{\alpha y_i}(x_2) \eta_i(x_2).$$

A problem can now be formulated in $x\eta$ -space in which the integral (5:5) is to be minimized in the class of admissible arcs $\eta_i = \eta_i(x)$ satisfying the differential equations (5:6) and the end-conditions (5:4). The minimizing arcs for this problem are all normal on every sub-interval in the sense of Section 1, since the transformed problem in xy -space has this normality, and satisfy the transversality conditions $\Omega_{\eta'_\alpha}(x_1) = 0$ where Ω denotes the expression $\Omega(x, \eta, \eta', \mu) = \omega + \mu_\alpha \Phi'_\alpha$. The differential equations of the extremals, the accessory equations, are

$$(5:7) \quad (d/dx)\Omega_{\eta'_i} - \Omega_{\eta_i} = 0, \quad \Phi'_\alpha = 0.$$

If η_i, μ_α is a solution of the accessory equations, then the values of the derivatives $\Omega_{\eta'_i}$ corresponding to these solutions may be designated as usual by ζ_i . The variables x, η_i, ζ_i are called canonical variables for the second variation.

The second definition of a focal point is then the following: a point 3 on E_{12} for which there exists a set of solutions $\eta_i = u_i(x), \zeta_i = z_i(x)$ of the accessory equations whose functions u_i satisfy the conditions $u_i(x_1) = u_i(x_3) = 0$ but are not identically

zero on x_1x_3 , and whose functions z_1 have $z_\alpha(x_1) = 0$, is called a focal point of the manifold (5:1) on E_{12} . The discussion following Theorem 5:1 implies that the solution η_1, ζ_1 of the accessory equations defining such a focal point on an arc E_{12} has all of its functions $\eta_1(x)$ vanishing at the value $x = x_1$.

In order to prove analytically the Necessary Condition of Mayer let us suppose that x_3 were the abscissa of a focal point between 1 and 2 on E_{12} , defined as above by a solution $\eta_1 = u_1; \zeta_1 = z_1$ of the accessory equations. An admissible arc for the second variation could then be determined by the conditions

$$(5:8) \quad \eta_1 = u_1(x) \text{ on } x_1 \leq x \leq x_3, \quad \eta_1 = 0 \text{ on } x_3 \leq x \leq x_2.$$

It is readily seen that the second variation of the integral I would vanish along this arc, for it would be (8, p. 726) the difference of the values at x_3 and x_1 of the sum $\eta_1 \zeta_1 = u_1 z_1$ which is zero at each of these points. The arc (5:8) would therefore have to minimize the second variation to which it gives the value zero. If it were such a minimizing arc, then, since it is normal, there would exist for it a unique set of multipliers $\mu_0 = 1, \mu_\alpha$ with which it would satisfy the equations

$$(5:9) \quad \begin{aligned} (d/dx)\Omega_{\eta_i}' - \Omega_{\eta_i} &= (d/dx)\omega_{\eta_i}' - \omega_{\eta_i} + \mu_\alpha' \phi_{\alpha y_i} = 0, \\ \phi_{\alpha y_i} \eta_i' + \phi_{\alpha y_i} \eta_i &= 0 \end{aligned}$$

on each of the sub-intervals x_1x_3 and x_3x_2 , and for which the variables $\zeta_1 = \Omega_{\eta_i}(x, \eta, \eta', \mu)$ would be continuous and satisfy at x_1 the transversality conditions $\zeta_\alpha = 0$. The continuity of the functions ζ_1 at $x = x_3$ is a consequence of the corner conditions for a minimizing arc. The equations (5:9) would also have the solution $\eta_1, \mu_\alpha(x) + c_\alpha$, where the c_α are arbitrary constants. In canonical variables the solution would have the

form $\eta_1(x)$, $\bar{\zeta}_1(x) = \zeta_1(x) + c_\alpha \phi_\alpha y_i$. On the interval $x_3 x_2$ the functions $\eta_1(x)$ vanish identically and $\bar{\zeta}_1(x) = \mu_\alpha \phi_\alpha y_i$. Hence if the constants c_α were taken to satisfy the relations $c_\alpha = -\mu_\alpha(x_2)$, the variables $\bar{\zeta}_1(x)$ would all vanish at x_2 , and the functions $\eta_1, \bar{\zeta}_1$ would be identically zero since the only solution of the accessory equations with elements all vanishing at x_2 is the identically zero solution. But this would imply a contradiction, for the functions $\eta_1 = u_1$ are by hypothesis not identically zero on the interval $x_1 x_3$. It follows therefore that if E_{12} is a minimizing arc for the original and transformed problems then there can be on it between 1 and 2 no focal point 3 of the manifold (5:1). This completes the analytical proof of the Necessary Condition of Mayer.

6. The Determination of Focal Points. In this section the determination of points 3 focal to the manifold

$$(6:1) \quad x_1 = \alpha_1, \quad y_r = \beta_{r1}$$

on a non-singular extremal arc E_{12} will be discussed for the cases when E_{12} is imbedded in four different types of families. For each family a matrix or determinant will be derived whose roots give the abscissas of the focal points.

The lemma below on certain properties of the solutions of the accessory equations (5:7) will be of use later in the section.

LEMMA 6:1. The only solution η_i, μ_α of the accessory equations satisfying the transversality conditions $\Omega \eta'_\alpha(x_1) = 0$ and having the functions η_i all identically zero on $x_1 x_3$, where $x_1 < x_3$, is the solution with $\eta_i \equiv \mu_\alpha \equiv 0$ on the interval.

For on this interval the accessory equations and transversality conditions for solutions η_i, μ_α with $\eta_i \equiv 0$ reduce to the simple expressions

$$\mu'_\alpha \phi_{\alpha y_i} = 0, \quad \mu_\alpha \phi_{\alpha y_\beta} = 0 \quad (\beta = 1, \dots, m).$$

By hypothesis the matrix $\|\phi_{\alpha y_i}\|$ has maximum rank along E_{12} and the determinant $|\phi_{\alpha y_\beta}|$ is different from zero at $x = x_1$. The multipliers μ_α thus have their derivatives all zero on $x_1 x_3$ and vanish at x_1 . They are therefore all identically zero on the interval.

The equivalence of the two definitions of focal point (see pages 15 and 20) will now be demonstrated by methods of Bliss (8, p. 727) and McFarlan (7, p. 191). It is well known that every solution $(\eta_1, \zeta_1) = (u_1, z_1)$ of the accessory equations in canonical variables is expressible in the form

$$u_1 = c_k y_{1a_k} + d_k y_{1b_k}, \quad z_1 = c_k v_{1a_k} + d_k v_{1b_k},$$

where the coefficients c_k and d_k are constants, and the derivatives $y_{1a_k}, y_{1b_k}, v_{1a_k}, v_{1b_k}$ are elements of the non-vanishing determinant

$$(6:2) \quad \begin{vmatrix} y_{1a_k} & y_{1b_k} \\ v_{1a_k} & v_{1b_k} \end{vmatrix}$$

of the $2n$ -parameter family of extremals of Section 3 in which E_{12} is imbedded.

The abscissas of the focal points of the manifold (6:1) on E_{12} are, according to the second definition, values of x for which the equations

$$(6:3) \quad \begin{aligned} u_1(x_3) &= c_k y_{1a_k}(x_3) + d_k y_{1b_k}(x_3) = 0, \\ u_1(x_1) &= c_k y_{1a_k}(x_1) + d_k y_{1b_k}(x_1) = 0, \\ z_\alpha(x_1) &= c_k v_{\alpha a_k}(x_1) + d_k v_{\alpha b_k}(x_1) = 0 \end{aligned}$$

have solutions c_k, d_k not all zero. These are precisely the values of x for which the determinant $\Delta(x, x_1, a, b)$ of equation (5:2)

vanishes. Conversely, whenever the determinant $\Delta(x, x_1, a, b)$ has a root $x_3 \neq x_1$, there exists a set of constants c_k, d_k , not all zero, for which u_1 and z_1 satisfy relations (6:3). If now the functions u_1 are identically zero on the interval $x_1 x_3$, then by Lemma 6:1 the multipliers ρ_α associated with them, and consequently the functions z_1 , also vanish identically on the interval. This is impossible since c_k and d_k are not all zero and determinant (6:2) has maximum rank. Hence the two definitions of focal point are equivalent.

THEOREM 6:1. Let E_{12} be an extremal arc which is contained in a $2n$ -parameter family of extremals of the transformed problem

$$\begin{aligned} y_i &= y_i(x, a_1, \dots, a_n, b_1, \dots, b_n), \\ \lambda_\alpha &= \lambda_\alpha(x, a_1, \dots, a_n, b_1, \dots, b_n) \\ &\quad (i = 1, \dots, n; \alpha = 1, \dots, m) \end{aligned}$$

for special values a_{10}, b_{10} of the parameters. Suppose that the determinant

$$\begin{vmatrix} y_{1a_k} & y_{1b_k} \\ v_{1a_k} & v_{1b_k} \end{vmatrix} \quad (k = 1, \dots, n)$$

of the family is different from zero at point 1 on E_{12} . Then the focal points 3 of manifold (6:2) on E_{12} are determined by the roots $x_3 \neq x_1$ of the determinant $\Delta(x, x_1, a_0, b_0)$ where

$$\Delta(x, x_1, a, b) = \begin{vmatrix} y_{1a_k}(x, a, b) & y_{1b_k}(x, a, b) \\ y_{ra_k}(x_1, a, b) & y_{rb_k}(x_1, a, b) \\ v_{\alpha a_k}(x_1, a, b) & v_{\alpha b_k}(x_1, a, b) \end{vmatrix} \quad (r = m + 1, \dots, n).$$

The second theorem of this section gives a criterion for the determination of focal points by means of a $2n - m$ parameter family of extremals.

THEOREM 6:2. If E_{12} is a member for parameter values a_v ($v = 1, \dots, 2n - m$) of a $(2n - m)$ -parameter family of extremals

$$y_i = y_i(x, a_1, \dots, a_{2n-m}), \quad v_i = v_i(x, a_1, \dots, a_{2n-m})$$

such that the determinant

$$(6:4) \quad \begin{vmatrix} y_{ia_v} & 0 \\ v_{ia_v} & \phi_{\alpha y_i} \end{vmatrix}$$

is different from zero at $x = x_1$, then the focal points 3 of the manifold (6:1) on E_{12} are determined by the roots $x_3 \neq x_1$ of the determinant $\Delta(x, x_1, a_0)$ where

$$\Delta(x, x_1, a) = \begin{vmatrix} y_{ia_v}(x, a) \\ y_{ra_v}(x_1, a) \end{vmatrix}.$$

To prove this theorem let x_3 be a root of the function $\Delta(x, x_1, a_0)$. Then since the determinant $|\phi_{\alpha y_\rho}|$ ($\rho = 1, \dots, m$) is by hypothesis different from zero at point 1, a set of constants c_v, d_α not all zero can be determined such that

$$(6:5) \quad \begin{aligned} c_v y_{ia_v}(x_3, a_0) &= 0, & c_v y_{ra_v}(x_1, a_0) &= 0, \\ c_v v_{\rho a_v}(x_1, a_0) + d_\alpha \phi_{\alpha y_\rho}(x_1, a_0) &= 0. \end{aligned}$$

The solution $\eta_i = c_v y_{ia_v}(x)$, $\zeta_i = c_v v_{ia_v}(x) + d_\alpha \phi_{\alpha y_i}$ of the accessory equations thus has $\eta_i(x_3) = \eta_r(x_1) = \zeta_\alpha(x_1) = 0$ and determines a focal point 3 of manifold (6:1) on E_{12} provided the functions η_i do not vanish identically on the interval $x_1 x_3$. This condition is satisfied. For if it were not then by Lemma 6:1 the multipliers μ_α and consequently the functions ζ_i would also

be identically zero on the interval $x_1 x_3$, and at point 1 either the determinant (6:4) or the constants c_α , d_α would vanish. Conversely, suppose the solution η_1, ζ_1 of the accessory equations defines a focal point 3 of manifold (6:1) on E_{12} . Since the determinant (6:4) is different from zero at point 1 on E_{12} , a set of constants c_α , d_α can be selected so that the equations

$$\begin{aligned}\eta_1(x_1) - c_\alpha y_{1\alpha_0}(x_1, a_0) &= 0, \\ \zeta_1(x_1) - c_\alpha v_{1\alpha_0}(x_1, a_0) - d_\alpha \phi_{\alpha y_1}(x_1, a_0) &= 0\end{aligned}$$

are satisfied, whence the functions $\eta_1 - c_\alpha y_{1\alpha_0}$, $\zeta_1 - c_\alpha v_{1\alpha_0} - d_\alpha \phi_{\alpha y_1}$ form a set of identically zero solutions of the accessory equations. The conditions which determine the focal point 3 are now expressible as in equations (6:5) which have solutions for c_α , d_α not all zero in case the determinant $\Delta(x_3, x_1, a_0)$ vanishes.

The following theorem on the determination of focal points by a $(2n - 2m)$ -parameter family of extremals satisfying the conditions $\phi_\alpha = 0$ will be of use in the second chapter of this paper.

THEOREM 6:3. Let E_{12} be a member for parameter values a_{ρ_0} ($\rho = 1, \dots, 2n - 2m$) of a $(2n - 2m)$ -parameter family of extremals

$$y_1 = y_1(x, a_1, \dots, a_{2n-2m}), \quad \lambda_\alpha = \lambda_\alpha(x, a_1, \dots, a_{2n-2m})$$

which satisfy the equations $\phi_\alpha = 0$ and for which the matrix

$$(6:6) \quad \left\| \begin{array}{cc} y_{1\alpha_\rho} & 0 \\ v_{1\alpha_\rho} & \phi_{\rho y_i} \end{array} \right\|$$

has rank $2n - 2m$ on E_{12} . Then the focal points 3 of the manifold (6:1) on E_{12} are the points where the matrix

$$(6:7) \quad \left\| \begin{array}{c} y_{1\alpha_\rho}(x) \\ y_{r\alpha_\rho}(x_1) \end{array} \right\|$$

has rank less than $2n - 2m$ on E_{12} .

This theorem follows readily by an argument similar to that in the proof of Theorem 6:2 if we show that every solution of the accessory equations satisfying the equations $\phi_{\alpha y_1} \eta_1 = 0$ is expressible linearly in terms of the columns of the matrix (6:6). The equations $\phi_{\alpha y_1} \bar{\eta}_1 = 0$ regarded as linear equations in $\bar{\eta}_1$, $\bar{\zeta}_1$ have exactly $2n - m$ linearly independent solutions. Such a system is the set of columns of (6:6). But η_1, ζ_1 is also such a solution. Hence constants c_ρ, d_α can be determined so that the equations

$$\begin{aligned}\eta_1(x_1) - c_\rho y_{1a_\rho}(x_1, a_0) &= 0, \\ \zeta_1(x_1) - c_\rho v_{1a_\rho}(x_1, a_0) - d_\alpha \phi_{\alpha y_1}(x_1, a_0) &= 0\end{aligned}$$

are satisfied, and the solution $\eta_1 - c_\rho y_{1a_\rho}, \zeta_1 - c_\rho v_{1a_\rho} - d_\alpha \phi_{\alpha y_1}$ of the accessory equations is therefore the identically zero solution.

An important criterion for the determination of focal points by means of an n -parameter family of extremals is given in

THEOREM 6:4. If E_{12} is a member for parameter values a_{k0} ($k = 1, \dots, n$) of an n -parameter family of extremals

$$y_1 = y_1(x, a_1, \dots, a_n), \quad v_1 = v_1(x, a_1, \dots, a_n)$$

satisfying the end-point and transversality conditions

$$(6:8) \quad y_r(x_1, a_1, \dots, a_n) = \beta_{r1}, \quad v_\alpha(x_1, a_1, \dots, a_n) = 0$$

and having the determinant

$$\begin{vmatrix} y_{\alpha a_k} \\ v_{ra_k} \end{vmatrix}$$

different from zero at (x_1, a_0) , then the focal points of the manifold (6:1) on E_{12} are determined by the roots $x_3 \neq x_1$ of the

function $D(x, a_0)$ where

$$D(x, a) = | y_{ia_k} |.$$

By differentiation of equations (6:8) with respect to a_k it is seen that the functions $y_{ra_k}, v_{\alpha a_k}$ all vanish at x_1 . If now x_3 is a root of determinant $D(x, a_0)$, constants c_k not all zero can be chosen so that the solution of the accessory equations

$$(6:9) \quad \eta_1 = c_k y_{ia_k}, \quad \zeta_1 = c_k v_{ia_k}$$

has its elements satisfying the conditions $\eta_1(x_3) = \eta_r(x_1) = \zeta_\alpha(x_1) = 0$ but η_1 not all identically zero on $x_1 x_3$. Conversely, let the solutions η_1, ζ_1 define a focal point 3 on E_{12} . Every solution η_1, ζ_1 of the accessory equations whose functions η_r, ζ_α vanish at x_1 is expressible in the form (6:9) where the c_k are suitably selected constants. The focal points 3 on E_{12} are then determined by values of x for which the equations

$$c_k y_{ia_k}(x_3, a_0) = 0$$

have solutions c_k not all zero, that is, by values x_3 which make the determinant $D(x_3, a_0)$ vanish.

7. The Sufficient Conditions. The definitions and notations of this section are those used by Bliss in the derivation of sufficient conditions for the problem of Lagrange with fixed end points (8, Chapter IV). Two lemmas will be deduced with whose help an n -parameter family of extremal arcs can be set up which form a field $\mathcal{F}$ containing as an extremal the arc E_{12} under consideration. It will be shown that when certain conditions are imposed upon E_{12} this arc furnishes a minimum for I for the original problem in the class of admissible arcs lying in a suitably restricted field $\mathcal{F}$ and passing through the end points of E_{12} .

A non-singular extremal arc E_{12} determines on the interval $x_1 - h \leq x \leq x_2$ an extended extremal which contains E_{12} as a part of it and is called an extension of E_{12} . For h sufficiently small and x at or between $x_1 - h$ and x_1 the determinant $|\Phi_{\alpha} y_s|$ ($\alpha, s = 1, \dots, m$) is different from zero on E_{12} . Hence at a point O of the extension of E_{12} having the coordinates $(x_0, y_{10}, \dots, y_{n0})$ with $x_1 - h \leq x_0 \leq x_1$ the end conditions

$$(7:1) \quad x = x_0, \quad y_r = y_{r0} \quad (r = m + 1, \dots, n)$$

with the equations $\Phi_{\alpha} = 0$ determine the remaining m coordinates $y_{\alpha} = y_{\alpha 0}$. When x_1 is replaced by x_0 in the definitions of a focal point in Section 5 and in Theorem 6:4, these become definitions of a focal point of the manifold (7:1) on the extended extremal E_{02} and a theorem on the determination of such focal points.

LEMMA 7:1. If h of the interval $x_1 - h \leq x \leq x_2$ on which the extension E_{02} is defined is taken sufficiently small, and if x_0 is a value on the interval $x_1 - h \leq x_0 < x_1$, then the conjugate system of solutions of the accessory equations u_{ik}, v_{ik} with initial values $u_{ik}(x_0) = \delta_{ik}, v_{ik}(x_0) = 0$ has the determinant $|u_{ik}|$ different from zero on the interval $x_1 - h \leq x \leq x_1 + h$. The Kronecker symbol δ_{ik} takes the value 1 when $i = k$ and vanishes for $i \neq k$.

To prove this it should first be noted that since E_{12} is a non-singular extremal there exists for it a set of linearly independent solutions of the accessory equations η_{is}, ζ_{is} , ($s = 1, \dots, 2n$), with a non-vanishing determinant at every value of x . The equations

$$(7:2) \quad \eta_{is}(\xi) c_{sk}(\xi) = \delta_{ik}, \quad \zeta_{is}(\xi) c_{sk}(\xi) = 0,$$

in which the $c_{sk}(\xi)$ are undetermined functions of ξ , have

therefore unique solutions for c_{sk} at each point ξ of the interval $x_1 - h \leq x \leq x_1$. The determinant $|\eta_{is}(x)c_{sk}(\xi)|$ becomes $|\delta_{ik}|$ at the values $x = x_1, \xi = x_1$. Because of the continuity of η_{is} and ζ_{is} it remains different from zero for $x_1 - h \leq x \leq x_1 + h, x_1 - h \leq \xi \leq x_1 + h$ when h is sufficiently small. Let ξ be taken at x_0 . Then the determinant $|\eta_{is}(x)c_{sk}(x_0)|$ becomes the identity for $x = x_0$ and does not vanish for x on the interval $x_1 - h, x_1 + h$. By equations (7:2) the corresponding determinant $|\zeta_{is}(x)c_{sk}(x_0)|$ has all of its elements zero at $x = x_0$. The conclusion of the lemma follows immediately when the conjugate system of solutions of the accessory equations is defined by the formulas

$$u_{ik} = \eta_{is}(x)c_{sk}(x_0), \quad v_{ik} = \zeta_{is}(x)c_{sk}(x_0).$$

It will now be shown, by procedure similar to that for the Lagrange problem (8, Section 32), that if condition III' is satisfied by E_{12} there is no focal point x_3 of manifold (7:1) on E_{02} on the interval $x_0 < x \leq x_1 + h$. The notations u_1, ρ_α, z_1 , and v_1, σ_α will be used for the linear expressions

$$\begin{aligned} u_1 &= a_k u_{1k}, & \rho_\alpha &= a_k \rho_{\alpha k}, & z_1 &= a_k v_{1k}, \\ v_1 &= a_k' u_{1k}, & \sigma_\alpha &= a_k' \rho_{\alpha k}, \end{aligned}$$

where u_{1k} and v_{1k} are elements of the conjugate system of solutions of Lemma 7:1, $\rho_{\alpha k}$ are the corresponding multipliers, and a_k are undetermined functions of x . Primes attached to expressions involving $u_1, v_1, \rho_\alpha, \sigma_\alpha$ will indicate derivatives with respect to x calculated as if the coefficients a_k, a_k' were constants. Let the functions $\eta_1(x)$ be a set of admissible variations along the arc E_{12} . The equations

$$\eta_1 = u_1 = a_k u_{1k}, \quad \rho_\alpha = a_k \rho_{\alpha k}$$

determine uniquely the coefficients a_k , and consequently the multipliers ρ_α , as functions of x on the interval $x_1 - h \leq x \leq x_1 + h$ on which the determinant $|u_{1k}|$ remains different from zero. If now η_1 belongs to a solution η_1, μ_α of the accessory equations it can be shown (8, p. 739) that for these multipliers μ_α associated with η_1 the x -derivative of the sum $\eta_1(\Omega \eta'_i - z_i)$ is the expression $F_{y_1' y_k'} (\eta_1' - u_1') (\eta_k' - u_k')$ which by III' is non-negative. Suppose that there were a focal point 3 of manifold (7:1) on $x_0 < x \leq x_1 + h$ defined by this set η_1, μ_α . Then the relations $\eta_1(x_3) = \eta_r(x_0) = \Omega \eta'_\alpha(x_0) = z_1(x_0) = 0$ would be satisfied and the sum $\eta_1(\Omega \eta'_i - z_i)$ would vanish at x_0 and x_3 . The functions η_1 would therefore be all identically zero on $x_0 x_3$ which is impossible according to the definition of a focal point. If condition IV' also is imposed upon the arc E_{12} , the non-vanishing on the interval $x_1 + h \leq x \leq x_2$ of the determinant $\Delta(x, x_1, a_0, b_0)$ of Theorem 6:1 and the continuity of this determinant in the variables x and x_0 imply the non-vanishing of the determinant $\Delta(x, x_0, a_0, b_0)$ on this same interval when x_0 is sufficiently close to x_1 . This completes the proof of

LEMMA 7:2. Let E_{12} be an extremal arc satisfying III' and IV'. Then there is a point 0 in the order 0, 1, 2 on the extension of E_{12} such that on E_{01} there is no focal point of the manifold

$$(7:1) \quad x = x_0, \quad y_r = y_{r0}.$$

By a modification of Theorem 6:4 in the substitution of x_0 for x_1 it is clear that the n -parameter family of extremal arcs

$$y_i = y_i(x, a_1, \dots, a_n), \quad v_i = v_i(x, a_1, \dots, a_n)$$

which is cut transversally by the manifold (7:1) has the determinant

$|y_{1a_k}|$ different from zero on the closed interval $x_1 \leq x \leq x_2$. The extremals therefore simply cover a neighborhood $\mathcal{F}$ of E_{12} in xy -space. In this neighborhood they form a field (8, p. 733) since on the initial manifold (7:1) the variables y_r are constants, the functions v_α are all zero, and the Hilbert integral is consequently independent of the path.

The results obtained thus far in Section 7 may be summarized in the following theorem, if it is noticed that an admissible arc without corners and satisfying I and III' is a non-singular extremal arc (8, p. 735):

THEOREM 7:1. If E_{12} is an admissible arc without corners satisfying I, III', IV' then there exists a field $\mathcal{F}$ containing E_{12} as one of its extremals.

The next theorem of this section gives an analogue of the integral formula of Weierstrass:

THEOREM 7:2. If the extremal arc E_{12} of Theorem 7:1 satisfies the end conditions $\psi_\mu = 0$ as given in equations (1:3), then the field $\mathcal{F}$ may be so restricted that for every admissible arc C_{52} in $\mathcal{F}$ with ends satisfying $\psi_\mu = 0$ the point 5 falls at 1 on E_{12} and the following relation holds

$$I(C_{52}) - I(E_{12}) = \int_{x_5}^{x_2} E[x, y, p(x, y), y'] dx.$$

In this formula the expression E is defined by the equality

$$E[x, y, p(x, y), y'] = f(x, y, y') - f(x, y, p) - (y_1' - p_1)f_{y_1}(x, y, p)$$

in which arguments $p_1(x, y)$ are slope functions of the field $\mathcal{F}$ and the integral is taken along the arc C_{52} .

The first conclusion of this theorem can be readily established as follows. The functions $\phi_\alpha(x, y)$ are all identically

zero along admissible arcs having end conditions $\psi_\mu = 0$, for all such arcs satisfy the equations $\phi'_\alpha = 0$ and pass through point 2 on E_{12} . Moreover because of the non-vanishing of the determinant $|\phi_\alpha y_\delta|$ at $x = x_1$ the relations $y_{r1} = \beta_{r1}$ of the set $\psi_\mu = 0$, and the equations $\phi_\alpha = 0$, suffice to determine the remaining m coordinates $y_{\alpha 1}$. The only solutions of these equations in a sufficiently small neighborhood M of the end-point 1 of E_{12} are the values $y_{\alpha 1} = \beta_{\alpha 1}$. The field $\mathcal{F}$ may be so restricted that the first of every pair of points $(x_1, y_1), (x_2, y_2)$ in it satisfying $\psi_\mu = 0$ lies in the neighborhood M (12, p. 270). Hence for every comparison arc C_{52} in $\mathcal{F}$ the first end-point 5 must lie at 1 on E_{12} . The relation

$$I^*(C_{52}) = I^*(E_{12}) = I(E_{12})$$

now justifies the final conclusion of the theorem as in the problem of Lagrange (8, p. 731).

Let the condition II_b' (8, p. 734) also be imposed on the arc E_{12} . The field $\mathcal{F}$ can then be taken so small that the elements $[x, y, p(x, y)]$ belonging to it all lie in the neighborhood of the elements (x, y, y') on E_{12} described in II_b' . The first sufficiency condition follows immediately, as stated in the theorem below.

SUFFICIENT CONDITIONS FOR A STRONG RELATIVE MINIMUM. If an admissible arc E_{12} without corners satisfies conditions I, II_b' , III' , IV' and the end conditions $\psi_\mu = 0$, then there is a neighborhood $\mathcal{F}$ of the points (x, y) on E_{12} such that the inequality $I(C_{52}) > I(E_{12})$ holds for every admissible arc which is in $\mathcal{F}$ and satisfies $\psi_\mu = 0$ but is not identical with E_{12} .

If condition II_b' is not assumed, then the following sufficiency theorem for a weak relative minimum can be deduced as in

the problem of Lagrange (8, p. 736).

SUFFICIENT CONDITIONS FOR A WEAK RELATIVE MINIMUM. If an admissible arc E_{12} without corners satisfies conditions I, III', IV', and the end conditions $\psi_\mu = 0$, then there is a neighborhood N of the sets of values (x, y, y') on E_{12} such that the inequality $I(C_{52}) > I(E_{12})$ holds for every admissible arc C_{52} whose elements are all in N and whose ends satisfy $\psi_\mu = 0$ but which is not identical with E_{12} .

Attention should perhaps be called to the fact that in these theorems the neighborhoods $\mathcal{F}$ and N should be restricted to lie in the region of equivalence of the original and transformed problems, so that the end-point 5 of every admissible arc C_{52} for the transformed problem falls at 1 on E_{12} . Then the conditions stated are sufficient to insure the minimizing properties of the arc E_{12} for the original problem as well as for the transformed one.

CHAPTER II

THE PROBLEM OF LAGRANGE WITH FINITE SIDE CONDITIONS
AS A SIMPLE PROBLEM

8. Introduction. As suggested in the introduction to this paper the problem of Lagrange with finite side conditions may be transformed into a simple problem of the calculus of variations with no side conditions in the following manner. The system of equations $\phi_\alpha = 0$ in the original formulation of the problem is enlarged by the adjunction (5, p. 312) of $n - m$ additional equations $\phi_r = z_r$ to have the form

$$(8:1) \quad \begin{aligned} \phi_\alpha(x, y) &= 0, & \phi_r(x, y) &= z_r \\ (\alpha &= 1, \dots, m; r = m + 1, \dots, n) \end{aligned}$$

where the z_r are new variables. The new functions $\phi_r(x, y)$ are so chosen that they have continuous derivatives of the first three orders and that the determinant $|\phi_{ky_1}|$ ($i, k = 1, \dots, n$) is different from zero in a neighborhood of the arc E_{12} under consideration. By means of the last $n - m$ relations of (8:1) the variables $y_1(x)$ on E_{12} determine functions $z_r(x)$ with like continuity properties. Implicit function theorems now imply that in a neighborhood of the elements (x, z) belonging to E_{12} , the equations (8:1) have solutions

$$(8:2) \quad y_1 = Y_1(x, z)$$

which are single-valued and continuous and have continuous derivatives with respect to x and z of the third order at least. Furthermore they reduce to the functions $y_1(x)$ on E_{12} , and their

derivatives with respect to x along an arc in xz -space are given by the formulas

$$(8:3) \quad Y_1' = Y_{1x} + Y_{1z_r} z_r'$$

in which primes denote total differentiation with respect to x . When the arguments y_1 and y_1' of the integrand f are replaced by Y_1 and Y_1' respectively, a new function g is obtained,

$$(8:4) \quad g(x, z, z') = f[x, Y(x, z), Y'(x, z)],$$

which has continuous derivatives up to and including those of the third order in its region of definition. Hence the new problem, equivalent to the original one, is that of minimizing the integral

$$J = \int_{x_1}^{x_2} g(x, z, z') dx$$

in a class of admissible arcs

$$(8:5) \quad z_r = z_r(x) \quad (x_1 \leq x \leq x_2; r = m+1, \dots, n)$$

satisfying the end conditions

$$(8:6) \quad \begin{aligned} x_1 &= \alpha_1, \quad z_r(x_1) = \phi_r[x_1, y(x_1)], \quad x_2 = \alpha_2, \\ z_r(x_2) &= \phi_r[x_2, y(x_2)]. \end{aligned}$$

This simple problem is equivalent to the original one at least in a neighborhood of E_{12} in which the equations (8:1) have solutions. Every minimizing arc (8:5) for it defines by (8:2) a minimizing arc $y_1 = y_1(x)$ for the original problem, and conversely every minimizing arc $y_1 = y_1(x)$ for the problem in xy -space determines by means of (8:1) an arc (8:5) with end values (8:6) which minimizes J .

The solutions $Y_1(x, z)$ of the system (8:1) satisfy the relations

$$(8:7) \quad \phi_{kx} + \phi_{ky_1} y_{1x} = 0, \quad \phi_{ky_1} y_{1z_r} = \delta_{kr}.$$

The matrices $\|\phi_{ky_1}\|$ and $\|\delta_{kr}\|$ have ranks n and $n - m$ respectively. Hence the matrix $\|y_{1z_r}\|$ is everywhere of rank $n - m$ (6, p. 51).

The relations below are listed for use later in the chapter. They are easily proved from the definition (8:4) of the function g .

$$(8:8) \quad g_{z_r} = f_{y_1} y_{1z_r} + f_{y_1'} y_{1z_r}', \quad g_{z_r}' = f_{y_1'} y_{1z_r}.$$

Furthermore along every arc $z_r = z_r(x)$ in xz -space differentiation of equations (8:7) with respect to x shows that

$$(8:9) \quad \phi_{ky_1'} y_{1z_r} + \phi_{ky_1} y_{1z_r}' = 0$$

is an identity in x .

9. The First Necessary Condition. In this section a first necessary condition on a minimizing arc for the original problem in xy -space will be derived by means of well-known results for the equivalent simple problem in xz -space.

For every minimizing arc E_{12} for the problem in xz -space there exists a set of constants e_r such that the equations

$$(9:1) \quad g_{z_r}' = \int_{x_1}^x g_{z_r} dx + e_r$$

are satisfied at every point of E_{12} .

With the help of formulas (8:8) the relations (9:1) are readily shown to be equivalent to the system

$$(9:2) \quad f_{y_1'} y_{1z_r} = \int_{x_1}^x (f_{y_1} y_{1z_r} + f_{y_1'} y_{1z_r}') dx + e_r,$$

which by use of DuBois-Reymond's partial integration (1, p. 27) is transformed into

$$(9:3) \quad \left[f_{y_1'} - \int_{x_1}^x f_{y_1} dx \right] y_{1z_r} = \int_{x_1}^x \left[f_{y_1'} - \int_{x_1}^x f_{y_1} dx \right] y_{1z_r}' dx + e_r.$$

If a set of constants K_1 is given, then a set of n multipliers $\nu_k(x)$, continuous except possibly at values of x defining corners of the minimizing arc may be determined such that at every point of E_{12} the equations

$$(9:4) \quad f_{y_1} - \int_{x_1}^x f_{y_1} dx = -\nu_k \phi_{ky_1} + \int_{x_1}^x \nu_k \phi_{ky_1}' dx + K_1$$

are satisfied. This determination can be made by the method described in reference (8, p. 680) applied to the function $F = f + \nu_k \phi_k'$. With the help of relations (8:7), (8:9) and (9:4) it now follows that the system (9:3) reduces to the simple form

$$(9:5) \quad -\nu_r(x) = -K_1 Y_{1z_r}(x_1) + e_r.$$

Since the matrix $\|Y_{1z_r}(x_1)\|$ has rank $n - m$, and in particular the determinant $|Y_{sz_r}(x_1)|$, ($s = m + 1, \dots, n$), does not vanish (9, p. 31), the first m of the quantities K_1 may be taken equal to zero and the remaining $n - m$ can be so determined that

$$(9:6) \quad -K_1 Y_{1z_r}(x_1) + e_r = 0.$$

The multipliers $\nu_r(x)$ consequently vanish and the conditions

$$F_{y_\alpha} = f_{y_\alpha} + \nu_\beta \phi_{\beta y_\alpha} = 0 \quad (\beta = 1, \dots, m)$$

hold at $x = x_1$. The multipliers $\nu_\alpha(x)$ are obviously uniquely determined.

The necessary condition above thus implies that for every minimizing arc for the original problem there exists a set of constants K_1 and a function $F = f + \nu_\alpha \phi_\alpha'$, in which the multipliers ν_α are continuous except possibly at values of x defining corners of E_{12} , such that equations

$$(9:7) \quad F_{y_1} = \int_{x_1}^x F_{y_1} dx + K_1$$

are satisfied at every point of E_{12} , and at the point 1 the derivatives F_{y_α} , all vanish.

Conversely, the First Necessary Condition satisfied by a minimizing arc in xz -space can be deduced from the condition just derived for the problem in xy -space. Because of the relations (8:7), (8:9), and

$$Y_{1z_r} \int_{x_1}^x F_{y_1} dx = \int_{x_1}^x \left[Y_{1z_r} F_{y_1} + Y_{1z_r}' \int_{x_1}^x F_{y_1} dx \right] dx$$

multiplication of equations (9:7) by Y_{1z_r} gives

$$f_{y_1}' Y_{1z_r} = \int_{x_1}^x \left\{ f_{y_1} Y_{1z_r} + \left[-\lambda_\alpha \phi_{y_\alpha} + \int_{x_1}^x F_{y_1} dx \right] Y_{1z_r}' \right\} dx + K_1 Y_{1z_r}.$$

Whence by (8:8), (9:6) and (9:7)

$$\begin{aligned} g_{z_r}' &= \int_{x_1}^x [g_{z_r} + K_1 Y_{1z_r}'] dx + K_1 Y_{1z_r} \\ &= \int_{x_1}^x g_{z_r} dx + e_r. \end{aligned}$$

Let the original and simple problems be considered in parametric form, as has been done by Bliss and Hestenes (15, p. If the first necessary condition is applied to this form, the results can be expressed in non-parametric form as in the Necessary Condition I of Section 2. It follows, therefore, that the first necessary condition for a minimum in xz -space implies and is implied by the necessary condition deduced above for the problem in xy -space.

Corollaries 1, 2 and 3 and their further implications as stated in Section 2 are immediate consequences of the necessary condition just derived in xz -space. In connection with Corollary 3 it is interesting to note that the determinant R vanishes if and only if the determinant

$$(9:9) \quad R_1 = | g_{z_r}, z_s' | \quad (r, s = m+1, \dots, n),$$

formed for the corresponding functions in xz -space, is also equal to zero. This is evident when the matrix product $\tilde{A}RA$ is formed where A is the matrix

$$A = \left\| \begin{array}{cc} \phi_{\alpha y_i} & y_{1z_r} \\ 0 & 0 \end{array} \right\| \quad (\alpha, \beta, \gamma = 1, \dots, m) \\ \delta_{\beta\gamma} \quad (r = m+1, \dots, n; i = 1, \dots, n)$$

in which the upper left-hand n 'th order minor is the reciprocal of the matrix $\|\phi_{ky_1}\|$ and $\tilde{A}$ is the transpose of A . A non-singular admissible arc $z_r = z_r(x)$ therefore determines by equations (8:2) a non-singular admissible arc for the original problem.

10. The Extremals. In this section it will be shown that an extremal arc in xz -space determines a corresponding extremal arc in xy -space. An imbedding theorem for extremals in xz -space will then be interpreted for extremal arcs in xy -space.

A set of functions $z_r(x)$ with continuous second derivatives determines, by means of solutions (8:2), a set $y_1(x)$ with like differentiability properties. The equations

$$(d/dx) g_{z_r} = g_{z_r}$$

defining an extremal arc in xz -space may therefore be transformed by formulas (8:8) into the system

$$(10:1) \quad (d/dx) f_{y_1} - f_{y_1} y_{1z_r} = 0.$$

It follows that equalities of the form

$$(10:2) \quad (d/dx) f_{y_1} - f_{y_1} = \Lambda_\alpha \phi_{\alpha y_i}$$

are satisfied by the coefficients of the y_{1z_r} in (10:1), since the rows of the matrix $\|\phi_{\alpha y_1}\|$ form m linearly independent solutions of the equations whose coefficients are the columns of the matrix $\|y_{1z_r}\|$ in terms of which all other such solutions are linearly

expressible. The multipliers Λ_α are continuous, for along an extremal the left members of equations (10:2) and the derivatives $\phi_{\alpha y_1}$ are continuous. Let new functions λ_α be given by the relations

$$\lambda_\alpha = - \int_{x_1}^x \Lambda_\alpha dx + b_\alpha$$

where the b_α are arbitrary constants of integration. By substitution from equations

$$(d/dx) \lambda_\alpha \phi_{\alpha y_i} = \lambda_\alpha \phi_{\alpha y_i}' - \Lambda_\alpha \phi_{\alpha y_i},$$

the system (10:2) becomes

$$(d/dx) F_{y_1} - F_{y_1} = 0$$

in which F is the expression $f + \lambda_\alpha \phi_\alpha'$. The conclusions of the theorem below are now evident, since the transversality conditions at $x = x_1$ have the form

$$F_{y_\alpha} = f_{y_\alpha} + \lambda_\beta \phi_{\beta y_\alpha} = 0 \quad (\beta = 1, \dots, m).$$

THEOREM 10:1. Every extremal arc in xz -space determines in xy -space an arc $y_1(x)$ having continuous derivatives of the first two orders, and having associated with it a set of continuous multipliers $\Lambda_\alpha(x)$ with which it satisfies the equations

$$(10:3) \quad (d/dx) f_{y_1} - f_{y_1} = \Lambda_\alpha \phi_{\alpha y_1}, \quad \phi_\alpha = 0.$$

If multipliers λ_α and a function F are defined by the formulas

$$(10:4) \quad \lambda_\alpha = - \int_{x_1}^x \Lambda_\alpha(x) dx + b_\alpha, \quad F = f + \lambda_\alpha \phi_\alpha',$$

then along the arc in xy -space the equations

$$(10:5) \quad (d/dx) F_{y_1} - F_{y_1} = 0, \quad \phi_\alpha = 0,$$

are also satisfied. The constants b_α may be so chosen that
 $F_{y_\alpha}(x_1) = 0.$

It is well known that every extremal arc for the original problem in xz -space along which the determinant R_1 (9:9) is different from zero is a member for parameter values $a_{\rho 0}$ ($\rho = 1, \dots, 2n - 2m$) of a $(2n - 2m)$ -parameter family of extremal arcs

$$z_r = z_r(x, a_1, \dots, a_{2n-2m}) \quad (x_1 \leq x \leq x_2)$$

whose functions z_r, z_{rx} have continuous partial derivatives of first order at least in a neighborhood of values (x, a) defining E_{12} , and for which the determinant

$$(10:6) \quad \begin{vmatrix} z_{ra_\rho} \\ z_{ra_\rho}' \end{vmatrix}$$

is different from zero on E_{12} .

By means of relations (8:2) and the definitions (10:3) of the multipliers Λ_α the functions of this family determine in xy -space a family of extremals

$$y_1 = y_1(x, a_1, \dots, a_{2n-2m}), \quad \Lambda_\alpha = \Lambda_\alpha(x, a_1, \dots, a_{2n-2m})$$

with like differentiability properties in x and a which is similar to the second family discussed in Section 3. When the multipliers Λ_α are substituted in formulas (10:4) they determine new multipliers λ_α as functions of x and the $2n - m$ arbitrary constants a_ρ and b_α possessing with their derivatives $\lambda_{\alpha x}$ continuous partial derivatives of first order at least in a neighborhood of values (x, a, b) on E_{12} . The functions $y_1(x, a)$, $\lambda_\alpha(x, a, b)$ thus define a $(2n - m)$ -parameter family of extremals in xy -space. If the arbitrary constants b_α are prescribed by the transversality conditions

$$F_{y_\alpha} = f_{y_\alpha} + \lambda_\delta \phi_\delta y_\alpha = 0 \quad (\delta = 1, \dots, m)$$

at $x = x_1$, then a family

$$y_1 = y_1(x, a_1, \dots, a_{2n-2m}), \quad \lambda_\alpha = \lambda_\alpha(x, a_1, \dots, a_{2n-2m})$$

depending upon only the $2n - 2m$ parameters a_ρ is derived. The matrix

$$(10:7) \quad \begin{vmatrix} y_{1a_\rho} & 0 \\ v_{1a_\rho} & \phi_{\alpha y_i} \end{vmatrix}$$

of this family, in which the functions v_1 are the derivatives F_{y_1} , has rank less than $2n - m$ if and only if the determinant (10:6) vanishes. For if the determinant (10:6) is zero there is a linear combination $z_{ra_\rho} c_\rho$, $v_{ra_\rho} c_\rho$ whose elements vanish identically, where x, z, v are the canonical variables for the xz -problem. With the help of the relations

$$\begin{aligned} 0 &= \phi_{\alpha y_1} y_{1a_\rho}, & v_r &= f_{y_1} y_{1z_r} = v_1 y_{1z_r}, \\ z_{ra_\rho} &= \phi_{ry_1} y_{1a_\rho}, & v_{ra_\rho} &= y_{1z_r} v_{1a_\rho} + y_{1z_r z_s} z_{sa_\rho} v_1, \end{aligned}$$

deduced from equations (8:1), (8:7), and (8:8), it is clear that

$$\phi_{ky_1} y_{1a_\rho} c_\rho = 0, \quad y_{1z_r} v_{1a_\rho} c_\rho = 0$$

from which it follows readily that

$$y_{1a_\rho} c_\rho = 0, \quad v_{1a_\rho} c_\rho = d_\alpha \phi_{\alpha y_i}$$

Hence the matrix (10:7) has rank less than $2n - m$. A similar procedure shows conversely that if the matrix (10:7) has rank less than $2n - m$ then the determinant (10:6) is zero.

11. The Necessary Conditions of Weierstrass, Legendre, and Jacobi. The Necessary Conditions of Weierstrass and Legendre for the simple problem in xz -space are readily interpreted for the

problem in xy -space in the form given in Section 4 of Chapter I. The interpretation of the Necessary Condition of Jacobi is made by use of results obtained in Sections 6 and 10 above.

According to the first of these conditions for the xz -problem the inequality

$$E(x, z, z', Z') = g(x, z, Z') - g(x, z, z') - (Z_r' - z_r')g_{z_r'}(x, z, z') \\ \geq 0$$

must be satisfied at every element (x, z, z') of a minimizing arc for every admissible set $(x, z, Z') \neq (x, z, z')$. By means of equations (8:8) this inequality takes the form

$$f(x, y, p) - f(x, y, y') - (p_1 - y_1')f_{y_1'}(x, y, y') \geq 0$$

where

$$p_1 = Y_{1x} + Y_{1z_r}Z_r', \quad y_1' = Y_{1x} + Y_{1z_r}z_r'.$$

If the former of these inequalities holds for all sets (x, z, z', Z') with (x, z, z') on E_{1z} and (x, z, Z') admissible, then the latter must hold for all sets (x, y, y', p) with (x, y, y') on E_{1z} and (x, y, p) admissible and satisfying the equations

$$\phi_{\alpha x} + \phi_{\alpha y_i} p_1 = 0.$$

This may be shown as follows. Every solution p_1 of these last equations satisfies also the relations

$$(11:1) \quad \phi_{\alpha y_1}(p_1 - Y_{1x}) = 0$$

as is seen with the help of the first n equations (8:7). The differences $p_1 - Y_{1x}$ must therefore have multipliers Z_r' by means of which they are expressed linearly in terms of the columns of the matrix $\|Y_{1z_r}\|$ in the form $p_1 - Y_{1x} = Y_{1z_r}Z_r'$, since the columns of this matrix are a complete system of linearly independent

solutions of the equations (11:1).

The Necessary Condition of Legendre for the simple problem requires that the inequality

$$g_{z_r, z_s} K_r K_s \geq 0 \quad (r, s = m+1, \dots, n)$$

shall be satisfied at every element (x, z, z') of a minimizing arc for every set K_r with elements not all zero. The left member of the inequality, equivalent by relations of Section 8 to the function $f_{y_1, y_k} Y_{1z_r} K_r Y_{kz_s} K_s$, reduces to the simple expression $f_{y_1, y_k} \pi_1 \pi_k$ when the sums $Y_{1z_r} K_r$ are replaced by new functions π_1 . Every set π_1 so determined satisfies the equations

$$\Phi_{\alpha y_i} \pi_i = 0,$$

and conversely every set π_1 satisfying these equations is expressible in the form $\pi_1 = Y_{1z_r} K_r$ since the rows of the matrix $\|Y_{1z_r}\|$ are a complete set of linearly independent solutions of equations (8:7). The values π_1 cannot all vanish since otherwise the quantities K_r would all be identically zero. There has thus been obtained

THE NECESSARY CONDITION OF CLEBSCH. At every element (x, y, y') of a minimizing arc for the problem of Lagrange with finite side conditions the inequality

$$f_{y_1, y_k} \pi_1 \pi_k \geq 0$$

must be satisfied by every set $(\pi_1, \dots, \pi_n) \neq (0, \dots, 0)$ which is a solution of the equations

$$\Phi_{\alpha y_i} \pi_i = 0.$$

In order to discuss the Necessary Condition of Jacobi, a conjugate point 1 on an extremal arc in xz -space must be shown to correspond to a focal point of the initial manifold

$$(11:2) \quad x'_1 = \alpha_1, \quad y_r = \beta_{r1}$$

on an extremal arc in xy -space. In Section 10 an extremal arc in xz -space along which the determinant R_1 is different from zero, and which is a member for parameter values $a_\rho = a_{\rho 0}$ ($\rho = 1, \dots, 2n - 2m$) of a $(2n - 2m)$ -parameter family of extremals

$$z_r = z_r(x, a_1, \dots, a_{2n-2m}) \quad (x_1 \leq x \leq x_2)$$

was seen to determine a non-singular extremal arc E_{12} in xy -space imbedded in a $(2n - 2m)$ -parameter family of extremals

$$y_1 = y_1(x, a_1, \dots, a_{2n-2m}), \quad \lambda_\mu(x, a_1, \dots, a_{2n-2m})$$

for which the matrix

$$\begin{vmatrix} y_{1a_\rho} & 0 \\ v_{1a_\rho} & \phi_{\rho y_i} \end{vmatrix}$$

has rank $2n - m$ along E_{12} . It follows readily from equations (10:8) that every root of

$$(11:3) \quad \begin{vmatrix} z_{ra_\rho}(x) \\ z_{ra_\rho}(x_1) \end{vmatrix}$$

is a point at which the matrix

$$(11:4) \quad \begin{vmatrix} y_{1a_\rho}(x) \\ y_{ra_\rho}(x_1) \end{vmatrix}$$

has rank less than $2n - 2m$. Hence with the help of Theorem 6:3 the Necessary Condition of Jacobi for the problem in xz -space is seen to imply the Necessary Condition of Mayer for the transformed

and original problems in xy -space, and conversely.

12. The Sufficient Conditions. Let the symbols I, II, III, IV denote the conditions for the problem in xz -space analogous to those similarly numbered for the original and transformed problems in xy -space. An admissible arc in xz -space without corners satisfying conditions I, II_p', III', IV' and the end conditions (8:6) is known to furnish a strong relative minimum for the integral

$$J = \int_{x_1}^{x_2} g(x, z, z') dx.$$

It has been shown in this chapter that these conditions are implied by conditions for the corresponding arc in xy -space identical with those in the hypothesis of Theorem 7:3. Hence we have here a new proof by means of the auxiliary problem in xz -space that the hypotheses of the sufficiency theorems of Section 7 are sufficient to insure their conclusions. The necessity of the conditions I, II, III, IV for a minimum for the problem in xy -space is similarly a consequence of the necessity of the analogous conditions in xz -space.

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